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Remarks on Concrete Orthomodular Lattices[1]

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An orthomodular lattice (OML) is called concrete if it is isomorphic to a collection of subsets of a set with partial ordering given by set inclusion, orthocomplementation given by set complementation, and finite orthogonal joins given by disjoint unions. Interesting examples of concrete OMLs are obtained by applying Kalmbach's construction $K(L)$ to an arbitrary bounded lattice L. This note provides several results regarding Kalmbach's construction, concrete OMLs, and the relationship between the notions. First, we provide order-theoretic and categorical characterizations of the OML $K(L)$ in terms of the bounded lattice L. Second, we provide an identity satisfied by each OML $K(L)$, but not valid in every concrete OML. This shows that the class of OMLs of the form $K(L)$ do not generate the variety of all concrete OMLs. Finally, we show that every concrete OML can be embedded into a concrete OML in which every element is a join of two or fewer atoms.

KEY WORDS: concrete; orthomodular lattice; Kalmbach's construction; variety.

Mathematics Subject Classifications (2000): 06C15, 81P10, 03G12.

1. INTRODUCTION

A collection $\mathcal{X}$ of subsets of a set X is called a class (Gudder, 1979), or partial field (Godowski, 1981), of sets if (i) $\emptyset$ and X belong to $\mathcal{X}$, (ii) $A \in \mathcal{X} \Rightarrow X - A \in \mathcal{X}$, and (iii) if $A, B \in \mathcal{X}$ and $A \cap B = \emptyset$, then $A \cup B \in \mathcal{X}$. A class of sets naturally forms an orthomodular poset (abbreviated: OMP) when equipped with the partial ordering of set containment and orthocomplementation of set complementation. An OMP is called concrete if it is isomorphic to one obtained from a class of sets. An orthomodular lattice (abbreviated: OML) is called concrete if it is concrete when considered as an OMP.

Primary results on concrete OMLs were obtained by Godowski (Godowski, 1981) in the early 1980s. Recall that a finitely additive state on an OML L is a map $s : L \to [0, 1]$ that satisfies $s(x \vee y) = s(x) + s(y)$ for each pair of elements x, y in L with $x \leq y'$. A state is called two-valued, or dispersion-free, if its range is $\{0, 1\}$. Godowski (1981) showed that an OML is concrete if, and

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only if, for any $x, y \in L$ with $x \not\leq y$ there is a two-valued state s with $s(x) = 1$ and $s(y) = 0$. This is often expressed by saying that the OML has a full set of two-valued states. An analogous result is easily seen to hold for concrete OMPs. More surprisingly, Godowski (1981) also showed that the class of concrete OMLs is closed under homomorphic images, subalgebras, and products, and therefore forms a variety of OMLs. To date, numerous studies have been made of concrete OMPs and OMLs (Godowski and Greechie, 1984; Ovchinnikov and Sultanbekov, 1998; Pták, 2000; Navara, 1993), and of varieties of OMLs defined through properties of their state spaces (Godowski, 1982; Mayet, 1985, 1986; Mayet and Pták, 2000).

At approximately the same time as Godowski's work, Kalmbach (1977) gave a method to construct an OML from a bounded lattice. Roughly, the idea is to glue together the Boolean algebras generated by the chains of a bounded lattice L to form an OML that we denote $K(L)$. The main application of this construction was to show that every lattice can be embedded into an OML, thereby showing that the variety of OMLs satisfies no non-trivial lattice identities. A number of other applications have also been found for this interesting construction (Harding, 1991; Kalmbach, 1983; Svozil, 1998). In one such investigation, Harding observed the basic property relating OMLs of the form $K(L)$ to the variety of concrete OMLs – every OML of the form $K(L)$ is concrete. The first complete proof of this fact was given by Mayet and Navara (1995). Kalmbach's construction can also be applied to a bounded poset P, producing a concrete OMP we denote $K(P)$.

Our purpose here is to establish several (somewhat unrelated) results about Kalmbach's construction, concrete OMLs, and the relationship between the notions. First, we provide order-theoretic and categorical characterizations of the OML $K(L)$. Second, we give an identity valid in all OMLs of the form $K(L)$, but not valid in all concrete OMLs. This shows that the OMLs of the form $K(L)$ do not generate the variety of concrete OMLs. Third, we show that every concrete OML can be embedded into a concrete OML in which each element is a join of two or fewer atoms.

This paper is organized in the following manner. The second section provides the basics of Kalmbach's construction, and the third gives our abstract characterizations of $K(L)$. The fourth section gives an identity valid in all OMLs of the form $K(L)$, but not in all concrete OMLs. The fifth, and final, section deals with embedding concrete OMLs into concrete atomic OMLs.

For general background on OMPs and OMLs the reader should consult (Kalmbach, 1983; Pták and Pulmannová, 1991).

2. KALMBACH'S CONSTRUCTION

In this section, we give the basic definition of the OML $K(L)$ constructed from a bounded lattice L. At the heart of matters lie the following well-known results about Boolean algebras generated by chains (Balbes and Dwinger, 1974, pp. 105–109).

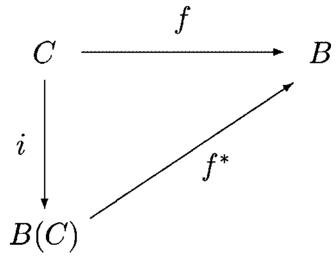
Theorem 1. *If C is a bounded chain, then up to isomorphism there is a unique Boolean algebra that contains C as a bounded subchain and is generated by C. Further, each element x of this Boolean algebra can be uniquely expressed as*

$$x = \bigvee_{i=1}^{n} x_{2i} \wedge x'_{2i-1}$$

where $x_1, \ldots, x_{2n}$ are elements of C and satisfy $x_1 < \cdots < x_{2n}$.

The Boolean algebra generated by a bounded chain C is usually called the free Boolean extension of C and will be denoted here as $B(C)$. This is a special case of a more general theory of free Boolean extensions of bounded distributive lattices (Balbes and Dwinger, 1974). A key result is that the free Boolean extension $B(D)$ of a bounded distributive lattice D is the reflective hull of D in the category of Boolean algebras. Specializing this result to bounded chains yields the following.

Theorem 2. *Suppose C is a bounded chain, $i : C \to B(C)$ is the natural embedding, B is a Boolean algebra, and $f : C \to B$ preserves bounds and order. Then there is a unique Boolean algebra homomorphism $f^* : B(C) \to B$ with $f^* \circ i = f$.*



By Theorem 1, there is a bijection between the elements of a Boolean algebra generated by a bounded chain C and the finite, even-length subchains of C. This allows for a synthetic construction of a Boolean algebra generated by C from the collection of such subchains of C. It is this idea that is exploited and generalized in Kalbach's construction.

Definition 3. For P a bounded poset, define

$$K(P) = \{x | x \text{ is a finite, even-length chain in } P\}.$$

Define a relation $\leq$ on $K(P)$ as follows. If $x = \{x_1, \ldots, x_{2n}\}$ and $y = \{y_1, \ldots, y_{2m}\}$ are elements of $K(P)$ with $x_1 < \cdots < x_{2n}$ and $y_1 < \cdots < y_{2m}$, set

$$x \leq y \Leftrightarrow \text{for each } i \leq n \text{ there is } j \leq m \text{ with } y_{2j-1} \leq x_{2i-1} < x_{2i} \leq y_{2j}.$$

Define a unary operation $\perp$ on $K(P)$ by setting $x^{\perp}$ to be the symmetric difference of the set x and the set $\{0, 1\}$.

Definition 4. For P a bounded poset define a map $i : P \to K(P)$ by setting

$$i(p) = \begin{cases} \{0, p\} & \text{if } p \neq 0 \\ \emptyset & \text{if } p = 0 \end{cases}$$

Remark As every bounded lattice L is also a bounded poset, we unambiguously use $K(L)$ to mean the application of the earlier construction to the bounded lattice L.

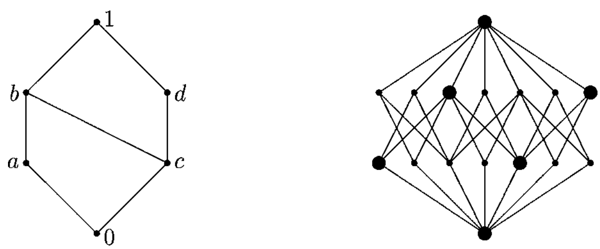
In Kalmbach's original paper (Kalmbach, 1977) this construction was considered only as it applied to a bounded lattice L. The following result, in the lattice setting, is due to Kalmbach (1977). A proof can be found either in her original paper, or using notation similar to the notation used here, in (Harding, 1991). We note that an alternate proof given by Kalmbach (1983) is flawed. The proof in the bounded poset setting is essentially a small fragment of the proof in the lattice setting.

Theorem 5. *If P is a bounded poset, then $K(P)$ is an OMP and $i : P \to K(P)$ is an order-embedding that preserves bounds. If L is a bounded lattice, then $K(L)$ is an OML and $i : L \to K(L)$ is a bounded lattice embedding.*

Remark For C a bounded chain, $K(C)$ is equal to the free Boolean extension $B(C)$ of C. However, if D is a bounded distributive lattice that is not a chain, one can show that $K(C)$ is not equal to the free Boolean extension $B(D)$ of D.

The following example may be instructive.

Example In the following diagram, a lattice L is at left, and the OML $K(L)$ is at right.



The least element of the OML $K(L)$ at right is the empty chain, and the greatest element is the chain $\{0, 1\}$. The atoms of $K(L)$, reading from left to right, are the chains $\{0, a\}$, $\{a, b\}$, $\{b, 1\}$, $\{c, b\}$, $\{0, c\}$, $\{c, d\}$, and $\{d, 1\}$; while the coatoms, again reading from left to right, are $\{a, 1\}$, $\{0, a, b, 1\}$, $\{0, b\}$, $\{0, c, b, 1\}$, $\{c, 1\}$, $\{0, c, d, 1\}$, and $\{0, d\}$. Note that the elements of $K(L)$ indicated by larger circles are the empty chain, $\{0, a\}$, $\{0, b\}$, $\{0, c\}$, $\{0, d\}$, and $\{0, 1\}$. These elements form a bounded sublattice of $K(L)$ that is isomorphic to L.

3. CHARACTERIZATIONS OF $K(L)$

In this section, we give abstract characterizations of $K(L)$ and $K(P)$. First, we provide an order-theoretic characterization of $K(L)$ along the lines of Theorem 1.

Theorem 6. *For L a bounded lattice and M an OML, M is isomorphic to $K(L)$ iff:*

1. *M has a bounded sublattice L' that is isomorphic to L.*
2. *For every block B of M, $B \cap L'$ is a chain that generates B.*

That $K(L)$ satisfies the earlier two conditions was established by Harding (1991). For the other implication, we assume that M is an OML that contains L as bounded sublattice, and that $B \cap L$ is a chain that generates B for each block B of M. We use a sequence of lemmas to show that M is isomorphic to $K(L)$. In the following we freely use results about commutativity when making computations in the OML M. All these results can be found in (Kalmbach, 1983).

Lemma 7. *For each x in M there is a least element $\overline{x}$ in L that lies above x.*

Proof: Note first that if x is an element of a Boolean algebra B that is generated by a chain C, then Theorem 1 shows that there is a least element of C lying above x. In fact, if $x = \vee_{i=1}^{n} x_{2i} \wedge x'_{2i-1}$, where $x_1 < \cdots < x_{2n}$ are elements of C, then x_{2n} is the least element of C lying above x.

Let $F = \{a \in L | x \leq a\}$ and note that F is a filter of L. Using Zorn's lemma, we can find a maximal chain C in F. Then $C \cup \{x\}$ is a set of pairwise commuting elements, hence is contained in a block B of M. Then, by our assumptions, $B \cap L$ is a chain D of L that generates B, and by construction, $C \subseteq D$.

As $x \in B$ and B is generated by a chain D, the earlier remarks show there is a least element $d \in D$ lying above x. Then $d \leq c$ for each $c \in C$, and as $d \in F$, it follows from the maximality of C that $d \in C$. Again using the maximality of C, and the fact that F is a filter in L, we have d is the least element of L lying above x. □

Lemma 8. *Suppose $x = a \wedge b'$ where $a, b \in L$ with $b < a$. Then $\overline{x} = a$.*

Proof: Let $\overline{x} = c$. Note that $a \in L$ and $x \leq a$ implies $\overline{x} \leq a$, hence $c \leq a$. Also, as $x \leq \overline{x}$ we have $a \wedge b' \leq c$. Taking the join of both sides of this expression with a' gives $b' \leq c \vee a'$. Thus, b commutes with $c \vee a'$ and clearly, b commutes with a. Therefore, b commutes with $a \wedge (c \vee a')$, and as $c \leq a$ this latter expression equals c.

So a, b, c are pairwise commuting, hence contained in some block B. As $x = a \wedge b'$ we have that x also belongs to B. Then as $B \cap L$ is a chain that generates B, our earlier comments show that a is the least member of $B \cap L$ lying above x. In particular, we have $a \leq c$. As we have already seen $c \leq a$, we have $a = c = \overline{x}$. □

Lemma 9. *Suppose $x_1, \ldots, x_{2n}$ are elements of L with $x_1 < \cdots < x_{2n}$. Then for $x = \vee_{i=1}^{n} x_{2i} \wedge x'_{2i-1}$ we have $\overline{x} = x_{2n}$.*

Proof: Surely $x \leq x_{2n}$ and therefore $\overline{x} \leq x_{2n}$. But $x_{2n} \wedge x'_{2n-1} \leq x$, and therefore $\overline{x_{2n} \wedge x'_{2n-1}} \leq \overline{x}$. It then follows from the previous lemma that $x_{2n} \leq \overline{x}$. □

Lemma 10. *Suppose $x_1, \ldots, x_{2n}$ and $y_1, \ldots, y_{2m}$ belong to L and that $x_1 < \cdots < x_{2n}$ and $y_1 < \cdots y_{2m}$ and $n \leq m$. Set*

$$x = \bigvee_{i=1}^{n} x_{2i} \wedge x'_{2i-1} \quad \textit{and} \quad y = \bigvee_{j=1}^{m} y_{2j} \wedge y'_{2j-1}.$$

If $x = y$, then $n = m$ and $x_i = y_i$ for each $1 \leq i \leq 2n$.

Proof: The proof is by induction on n. If $n = 0$ then x is the join of the empty family, hence $x = 0$. But if $m > 0$ then as $y_1 < y_2$ the orthomodular law provides $y_2 \wedge y'_1 \neq 0$, a contradiction. Thus, $m = 0$ and the claim is verified.

Suppose $n \geq 1$, and therefore $m \geq 1$. As $x = y$ we have $\overline{x} = \overline{y}$, and therefore by the previous result that $x_{2n} = y_{2m}$. It follows that $x_{2n} \wedge x' = y_{2m} \wedge y'$. Note

if $n = 1$	and	$x_1 = 0$	then	$x_{2n} \wedge x' = 0,$
if $n = 1$	and	$x_1 \neq 0$	then	$x_{2n} \wedge x' = x_1 \wedge 0',$
if $n > 1$	and	$x_1 = 0$	then	$x_{2n} \wedge x' = (x_{2n-1} \wedge x'_{2n-2}) \vee \cdots \vee (x_3 \wedge x'_2),$
if $n > 1$	and	$x_1 \neq 0$	then	$x_{2n} \wedge x' = (x_{2n-1} \wedge x'_{2n-2}) \vee \cdots \vee (x_1 \wedge 0').$

Of course, similar statements apply to $y_{2m} \wedge y'$.

In any of the earlier cases, the previous lemma (or the trivial fact that $\overline{0} = 0$) gives that $\overline{x_{2n} \wedge x'} = x_{2n-1}$. As $x_{2n} \wedge x' = y_{2m} \wedge y'$ we then have $x_{2n-1} = y_{2m-1}$.

Therefore, $x \wedge (x_{2n} \wedge x'_{2n-1})' = y \wedge (y_{2m} \wedge y'_{2m-1})'$. Note

$$x \wedge (x_{2n} \wedge x'_{2n-1})' = (x_{2n-2} \wedge x'_{2n-3}) \vee \cdots \vee (x_2 \wedge x'_1),$$
$$y \wedge (y_{2m} \wedge y'_{2m-1})' = (y_{2m-2} \wedge y'_{2m-3}) \vee \cdots \vee (y_2 \wedge y'_1).$$

with the understanding that the right side of either of these equations could be the join of the empty family (if $n = 1$ or $m = 1$) and therefore be zero.

The inductive hypothesis gives $n - 1 = m - 1$ and $x_i = y_i$ for each $1 \leq i \leq 2(n-1)$. Therefore, $n = m$, and as we have shown $x_{2n} = y_{2m}$ and $x_{2n-1} = y_{2m-1}$, we have $x_i = y_i$ for each $1 \leq i \leq 2n$. □

Definition 11. Define $\Gamma : K(L) \to M$ by setting

$$\Gamma(x) = \bigvee_{i=1}^{n} x_{2i} \wedge x'_{2i-1} \quad \text{if } x = \{x_1, \ldots, x_{2n}\} \text{ where } x_1 < \cdots < x_{2n}.$$

Note the operations on the right of the equality sign are operations in the OML M applied to the elements $x_1, \ldots, x_{2n}$ which belong to L. Note also that $\Gamma(\emptyset)$ is the join of the emptyset, hence equal to zero.

Theorem 12. *The map $\Gamma : K(L) \to M$ is an isomorphism.*

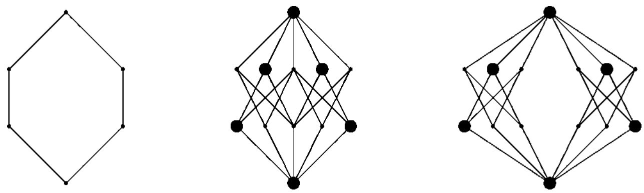
Proof: The map Γ is clearly well defined and the previous lemma shows Γ is one-to-one. Suppose $a \in M$. Then there is a block B of M that contains the element a. As $B \cap M$ is a chain that generates B, by Theorem 1 we can find $x_1 < \cdots < x_{2n}$ in this chain with $a = \bigvee_{i=1}^{n} x_{2i} \wedge x'_{2i-1}$. Setting $x = \{x_1, \ldots, x_{2n}\}$ we have $\Gamma(x) = a$. Thus, Γ is onto, hence a bijection.

Suppose that $x, y \in K(L)$ with $x = \{x_1, \ldots, x_{2n}\}$ and $y = \{y_1, \ldots, y_{2m}\}$ where $x_1 < \cdots < x_{2n}$ and $y_1 < \cdots < y_{2m}$. If $x \leq y$ in $K(L)$ then for each $1 \leq i \leq n$ there is $1 \leq j \leq m$ with $y_{2j-1} \leq x_{2i-1} < x_{2i} \leq y_{2j}$. But this implies that in M we have $x_{2i} \wedge x'_{2i-1} \leq y_{2j} \wedge y'_{2j-1}$, and therefore that $\Gamma(x) \leq \Gamma(y)$. So Γ is order preserving.

Conversely, working with the same elements x, y as in the previous paragraph, suppose $\Gamma(x) \leq \Gamma(y)$. Then $\Gamma(x)$ and $\Gamma(y)$ commute, and therefore belong to some block B of M. Let $C = B \cap M$ and note that by assumption C is a chain that generates B. By the uniqueness of the representations of x and y in terms of chains of elements of L given by the previous lemma, we have that $x_1, \ldots, x_{2n}$ and $y_1, \ldots, y_{2m}$ must all belong to C. Having reduced matters to the Boolean setting, it then follows easily, and is well known (Balbes and Dwinger, 1974), that for each $1 \leq i \leq n$ there is $1 \leq j \leq m$ with $y_{2j-1} \leq x_{2i-1} < x_{2i} \leq y_{2j}$. Therefore, $x \leq y$, showing that Γ is an order-isomorphism, and hence a bounded lattice isomorphism.

Finally, that $\Gamma(x') = \Gamma(x)'$ is a simple computation based on the definition of complementation in $K(L)$. □

Remark Theorem 6 does not generalize directly to the setting of bounded posets as is seen by the following example.

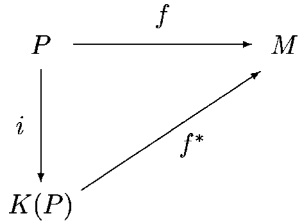


The figure at left is the bounded poset P. The figure at right is $K(P)$, with the large dots indicating how P sits inside. The figure in the middle is an OMP M that contains a bounded subposet P' (indicated by large dots) that is isomorphic to P. Note that each block B of M intersects P' in a chain that generates B, yet M is not isomorphic to $K(P)$. The trouble begins with the failure of Lemma 7 as the atom in the center of M has no least element of P' lying above it. Perhaps Theorem 6 could be generalized to the setting of bounded posets by including the regularity (Harding, 1998; Pták and Pulmannová, 1991) of M as well as the conclusion of Lemma 10 as assumptions. We have not investigated this question thoroughly as it lies somewhat outside of our interests.

We next turn our attention to a categorical characterization of $K(P)$ along the lines of Theorem 2. We first need to introduce some terminology.

Definition 13. An OMP-homomorphism is a map $f : M \to Q$ between OMPs that preserves bounds, orthocomplementation, order, and finite orthogonal joins.

Theorem 14. *Let P be a bounded poset, M be an OMP, $f : P \to M$ preserve bounds and order, and $i : P \to K(P)$ be as in Definition 4.. Then there exists a unique OMP-homomorphism $f^* : K(P) \to M$ with $f^* \circ i = f$.*



Proof: Suppose C is a bounded subchain of P. Then C generates a Boolean subalgebra of $K(P)$ and this Boolean algebra is literally equal to $K(C)$. Note that the restriction $f|C$ preserves bounds and order. As the image of $f|C$ is a chain of M, it generates a Boolean subalgebra B of M. Therefore, by Theorem 2 there is a unique Boolean algebra homomorphism $(f|C)^* : K(C) \to B$ with $(f|C)^* \circ (i|C) = f|C$. One sees easily that $(f|C)^*$ is in fact the unique OMP-homomorphism from $K(C)$ to M with this property.

Suppose C and D are bounded subchains of P and that x belongs to both $K(C)$ and $K(D)$. Then $x = \{x_1, \ldots, x_{2n}\}$ for some family of elements $x_1 < \cdots < x_{2n}$ belonging to both C and D. As $(f|C)^*$ is a Boolean algebra homomorphism satisfying $(f|C)^* \circ (i|C) = f|C$, we have $(f|C)^*(x) = \vee_{i=1}^{n} f(x_{2i}) \wedge f(x_{2i-1})'$. As a similar comment applies to $(f|D)^*$ we have that $(f|C)^*$ and $(f|D)^*$ agree on the intersection of their domains.

Consider functions as sets of ordered pairs. As each element of $K(P)$ belongs to $K(C)$ for some bounded subchain C of P, the remarks of the previous paragraph show that $f^* = \cup\{(f|C)^* | C \text{ is a bounded subchain of } P\}$ is a well-defined function from $K(P)$ to M. Clearly, f^* preserves bounds and orthocomplementation. Suppose that x, y belong to $K(P)$. If $x \leq y$ then there is a bounded subchain C of P with x, y belonging to $K(C)$. It then follows that $(f|C)^*(x) \leq (f|C)^*(y)$ and therefore that $f^*(x) \leq f^*(y)$. Similarly, if x, y are orthogonal, then $f^*(x \vee y) = f^*(x) \vee f^*(y)$. We therefore have that f^* is an OMP-homomorphism.

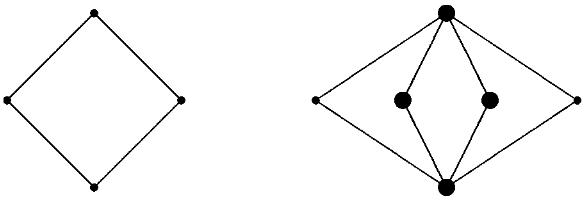
As $(f|C)^* \circ (i|C) = f|C$ for each bounded subchain C of P, it follows that $f^* \circ i = f$. Suppose that $g : K(P) \to M$ is an OMP-homomorphism with $g \circ i = f$. If C is a bounded subchain of P then the image of C under f generates a Boolean subalgebra B of M. We easily see that $g|K(C)$ is a Boolean algebra homomorphism from $K(C)$ into B and that $(g|K(C)) \circ (i|C) = f|C$. It follows from Theorem 2 that $g|K(C)$ is equal to $(f|C)^*$ for each bounded subchain C of P, hence $g = f^*$. □

Definition 15. Let **POS** be the category of bounded posets whose morphisms are order-preserving maps that preserve bounds. Let **OMP** be the category of OMPs whose morphisms are OMP-homomorphisms.

By general considerations, the following is a direct consequence of Theorem 14.

Theorem 16. *Kalmbach's construction provides a functor* $\mathbf{K} : \mathbf{POS} \to \mathbf{OMP}$ *that is left-adjoint to the functor* $\mathbf{U} : \mathbf{OMP} \to \mathbf{POS}$ *that forgets orthocomplementation.*

Remark If we consider the category **OML** of OMLs and OML-homomorphisms, and the category **LAT** of lattices and lattice homomorphisms, the situation in regards to Kalmbach's construction does not work out so nicely. Indeed, Kalmbach's construction does not even provide a functor from **LAT** to **OML** as is seen by the following example.



There is a bounded lattice homomorphism from the lattice L, depicted at the left, to the two-element lattice **2**. But there is no OML-homomorphism from $K(L)$, depicted at right, to $K(\mathbf{2}) = \mathbf{2}$ as $K(L)$ is the OML usually called MO_2, which is simple.

Of course, the functor from **OML** to **LAT** that forgets orthocomplementation does have a left-adjoint – and this is the functor that associates to a lattice L the OML freely generated by L. But the OML freely generated by L is not given by $K(L)$, and a description of its structure is almost a completely open question.

One can however salvage something from the previous theorem in the setting of OMLs. As the application of Kalmbach's construction to a bounded lattice L yields an OML $K(L)$, the earlier result specializes to show Kalmbach's construction provides a left-adjoint to the functor from the category of lattices and lattice homomorphisms to the category of OMLs and OMP-homomorphisms that forgets orthocomplementation.

4. AN IDENTITY

In this section we provide a identity valid in each OML $K(L)$ but not valid in all concrete OMLs. This identity may provide insight into the structure of OMLs $K(L)$.

Definition 17. For elements x, y of an OML L define the commutator $com(x, y)$ by

$$com(x, y) = (x \vee y) \wedge (x \vee y') \wedge (x' \vee y) \wedge (x' \vee y').$$

For elementary properties of commutators see (Kalmbach, 1983).

We say a and b are comparable if either $a \leq b$ or $b \leq a$. We write $a \sim b$ to indicate a, b are comparable and $a \not\sim b$ to indicate they are incomparable.

Proposition 18. *For* $x, y \in K(L)$ *with* $x = \{x_1, \ldots, x_{2n}\}$ *and* $y = \{y_1, \ldots, y_{2m}\}$

$$com(x, y) = \bigvee\{\{x_i \wedge y_j, x_i \vee y_j\}|\ 1 \leq i \leq 2n, 1 \leq j \leq 2m \text{ and } x_i \not\sim y_j\}.$$

Proof: Set $w = \bigvee\{\{x_i \wedge y_j, x_i \vee y_j\}|1 \leq i \leq 2n, 1 \leq j \leq 2m$ and $x_i \not\sim y_j\}$ and suppose $w = \{w_1, \ldots w_{2r}\}$ with $w_1 < \cdots < w_{2r}$.

Suppose $x_i \not\sim y_j$. If $x \vee y = \{z_1, \ldots, z_{2t}\}$ with $z_1 < \cdots < z_{2t}$ then there are p, q with $z_{2p-1} \leq x_i \leq z_{2p}$ and $z_{2q-1} \leq y_j \leq z_{2q}$. But $x_i \not\sim y_j$ so $p = q$, giving $z_{2p-1} \leq x_i \wedge y_j < x_i \vee y_j \leq z_{2p}$. Thus, $\{x_i \wedge y_j, x_i \vee y_j\} \leq x \vee y$. A similar argument shows $\{x_i \wedge y_j, x_i \vee y_j\} \leq x' \vee y, x \vee y', x' \vee y'$, and therefore that $\{x_i \wedge y_j, x_i \vee y_j\} \leq com(x, y)$. This shows that $w \leq com(x, y)$.

For the other inequality, we first show that $x \cup w$ and $y \cup w$ are chains. If x_i is incomparable to some y_j, then for some p we have $w_{2p-1} \leq x_i \wedge y_j < x_i \vee y_j \leq w_{2p}$. So if x_i is incomparable to some member of y, then x_i is comparable to each member of w. If x_i is comparable to each element of y, then as x_i is comparable to each member of x, it follows that x_i is comparable to each $x_k \wedge y_j$ and to each $x_k \vee y_j$. As the elements of w belong to the sublattice generated by the $x_k \wedge y_j$ and $x_k \vee y_k$ (Harding, 1991), it follows that x_i is comparable to each member of w. Therefore, $x \cup w$ is a chain, and by symmetry, $y \cup w$ is a chain. Therefore, x and y commute with w.

As w commutes with x, y it also commutes with everything in the subalgebra generated by x, y (Kalmbach, 1983). Therefore, $com(x, y) = (com(x, y) \wedge w) \vee (com(x, y) \wedge w') = com(x \wedge w, y \wedge w) \vee com(x \wedge w', y \wedge w')$.

As $x \cup w$ is a chain, the elements of $x \wedge w'$ all belong to $x \cup w$, and similarly the elements of $y \wedge w'$ all belong to $y \cup w$. We claim that all the elements of $x \wedge w'$ are comparable to all the elements of $y \wedge w'$, so $(x \wedge w') \cup (y \wedge w')$ is a chain. As $x \cup w$ and $y \cup w$ are chains, it is sufficient to show that if $x_i \not\sim y_j$ then either x_i is not an element of $x \wedge w'$ or y_j is not an element of $y \wedge w'$. But $x_i \not\sim y_j$ implies $x_i \wedge y_j < x_i, y_j < x_i \vee y_j$, and as $\{x_i \wedge y_j, x_i \vee y_j\} \leq w$, this shows that x_i does not belong to $x \wedge w'$ and y_j does not belong to $y \wedge w'$.

As all elements of $x \wedge w'$ are comparable to all elements of $y \wedge w'$ we have that $x \wedge w'$ and $y \wedge w'$ commute, hence $com(x \wedge w', y \wedge w') = 0$. Thus, from the earlier remarks, $com(x, y) = com(x \wedge w, x \wedge y)$. As $com(x \wedge w, y \wedge w) \leq (x \wedge w) \vee (y \wedge w) \leq w$ we have $com(x, y) \leq w$ as required. □

Definition 19. Let $x, y \in K(L)$. Call $x_i \in x$ and $y_j \in y$ maximally incomparable if x_i and y_j are incomparable, x_i is the maximal member of x that is incomparable to y_j, and y_j is the maximal member of y that is incomparable to x_i. Minimally incomparable elements are defined dually.

Proposition 20. *If* $x, y \in K(L)$*, then each member of* $com(x, y)$ *may be expressed as the join of a maximally incomparable pair of elements of* x, y *or as the meet of a minimally incomparable pair of elements of* x, y.

Proof: Suppose $x = \{x_1, \ldots, x_{2n}\}$ where $x_1 < \ldots < x_{2n}$, $y = \{y_1, \ldots, y_{2m}\}$ where $y_1 < \cdots < y_{2m}$ and $com(x, y) = \{w_1, \ldots, w_{2r}\}$ where $w_1 < \cdots < w_{2r}$.

Consider the relation $\not\sim$ of incomparability on $x \cup y$. For any element x_i that is incomparable to some member of y define x_i^+ and x_i^- to be the largest and least members of x that are related to x_i in the transitive closure $\not\sim^*$ of $\not\sim$. If y_j is incomparable to some member of x we define y_j^+ and y_j^- similarly.

Suppose $x_i \not\sim y_j$. Define $i_0 = i$, $j_0 = j$. Let x_{i_1} be the largest member of x that is incomparable to y_{j_0} and note $x_{i_0} \leq x_{i_1}$. let y_{j_1} be the maximal member of y with y_{j_1} incomparable to x_{i_1} and note $y_{j_0} \leq y_{j_1}$. Let x_{i_2} be maximal in x with x_{i_2} incomparable to y_{j_1} and so forth. This produces two increasing sequences $x_{i_0} \leq x_{i_1} \leq x_{i_2} \leq \cdots$ and $y_{j_0} \leq y_{j_1} \leq y_{j_2} \leq \cdots$. As x, y are finite chains, these sequences eventually stabilize, and one can see that they stabilize at x_i^+ and y_j^+. This implies that x_i^+ and y_j^+ are maximally incomparable. One similarly sets $x_{i_{-1}}$ to be least in x incomparable to y_{j_0} and so forth to produce sequences $x_{i_0} \geq x_{i_{-1}} \geq x_{i_{-2}} \geq \cdots$ and $y_{j_0} \geq y_{j_{-1}} \geq y_{j_{-2}} \geq \cdots$ that stabilize at the minimally incomparable pair x_i^- and y_j^-.

It follows from Proposition 18 that for each pair of incomparable elements x_s, y_t there is some p with $w_{2p-1} \leq x_s, y_t \leq w_{2p}$. As x_{i_0}, y_{j_0} are incomparable, there is p with $w_{2p-1} \leq x_{i_0}, y_{j_0} \leq w_{2p}$. As x_{i_1} and y_{i_0} are also incomparable, they are both are bounded by w_{2p-1} and w_{2p}. As x_{i_1} and y_{j_1} are incomparable, they also are bounded by w_{2p-1} and w_{2p}. In this manner we obtain that each member of the sequences $\ldots x_{i_{-1}}, x_{i_0}, x_{i_1}, \ldots$ and $\ldots, y_{j_{-1}}, y_{j_0}, y_{j_1}, \ldots$ are bounded below by w_{2p-1} and above by w_{2p}. In particular, $w_{2p-1} \leq x_i^-, y_j^-$ and $x_i^+, y_j^+ \leq w_{2p}$. This then yields that $\{x_i \wedge y_j, x_i \vee y_j\} \leq \{x_i^- \wedge y_j^-, x_i^+ \vee y_j^+\} \leq com(x, y)$.

It follows from Proposition 18 and the remarks in the preceding paragraph that $com(x, y) = \bigvee\{\{x_i^- \wedge y_j^-, x_i^+ \vee y_j^+\} | x_i \not\sim y_j\}$. As x_i^+, y_j^+ are maximally incomparable we have that $x_i^+ \vee y_j^+$ is comparable to each element of x and y, and as x_i^-, y_i^- are minimally incomparable we have $x_i^- \wedge x_j^-$ is comparable to each element of x, y. Therefore, the set of all elements of the form $x_i^- \wedge y_j^-$ or $x_i^+ \vee y_j^+$ form a chain. From the earlier description of $com(x, y)$ it follows that all the elements in $com(x, y)$ are in the sublattice generated by this chain (Harding, 1991), hence belong to this chain, and therefore are of the form $x_i^+ \vee y_j^+$ or $x_i^- \wedge y_j^-$. □

We shall require several technical lemmas. In each of these lemmas we assume that x, y, z are elements of $K(L)$ with $x = \{x_1, \ldots, x_{2n}\}$ where $x_1 < \ldots < x_{2n}$, $y = \{y_1, \ldots, y_{2m}\}$ where $y_1 < \ldots < y_{2m}$, and $z = \{z_1, \ldots, z_{2u}\}$ where $z_1 < \ldots < z_{2u}$.

Lemma 21. *Assume (i) x_i, y_j are maximally incomparable in x, y, (ii) $x_i \vee y_j$ is an element of $com(x, y)$, (iii) $x_i \vee y_j, z_k$ are maximally incomparable in $com(x, y)$, z, and (iv) x_p, z_q are maximally incomparable in x, z. Then $x_i \vee y_j \vee z_k \sim x_p \vee z_q$.*

Proof: Consider several cases. If $q < k$ then as x_p, z_q are maximally incomparable, $x_p \leq z_k$, so $x_p \vee z_q \leq z_k \leq x_i \vee y_j \vee z_k$. If $k < q$ then as $x_i \vee y_j, z_k$ are maximally incomparable $x_i \vee y_j \leq z_q$, so $x_i \vee y_j \vee z_k \leq z_q \leq x_p \vee z_q$. If $q = k$ and $i < p$ then as x_i, y_j are maximally incomparable, $y_j \leq x_p$, so $x_i \vee y_j \leq x_p$, giving $x_i \vee y_j \vee z_k \leq x_p \vee z_q$. If $q = k$ and $p \leq i$ then $x_p \leq x_i$, so $x_p \vee z_q \leq x_i \vee y_j \vee z_k$. □

Lemma 22. *Assume (i) x_i, y_j are maximally incomparable in x, y, (ii) $x_i \vee y_j$ is an element of com(x, y), (iii) $x_i \vee y_j, z_k$ are maximally incomparable in com$(x, y), z$, and (iv) x_p, z_q are minimally incomparable in x, z. Then $x_i \vee y_j \vee z_k \sim x_p \wedge z_q$.*

Proof: Consider several cases. If $p \leq i$ then $x_p \leq x_i$ so $x_p \wedge z_q \leq x_i \vee y_j \vee z_k$. If $q \leq k$ then $z_q \leq z_k$ so $x_p \wedge z_q \leq x_i \vee y_j \vee z_k$. Assume $i < p$ and $k < q$. As $i < p$ and x_i, y_j are maximally incomparable, we have $y_j \leq x_p$, hence $x_i \vee y_j \leq x_p$. Also, as $k < q$ and x_p, z_q are minimally incomparable, we have $z_k \leq x_p$. Combining these observations gives $x_i \vee y_j \vee z_k \leq x_p$. But as $k < q$, $x_i \vee y_j$ and z_k being maximally incomparable gives $x_i \vee y_j \vee z_k \leq z_q$. Thus, $x_i \vee y_j \vee z_k \leq x_p \wedge z_q$. □

Lemma 23. *Assume (i) x_i, y_j are minimally incomparable in x, y, (ii) $x_i \wedge y_j$ is an element of com(x, y), (iii) $x_i \wedge y_j, z_k$ are maximally incomparable in com$(x, y), z$, and (iv) x_p, z_q are maximally incomparable in x, z. Then $(x_i \wedge y_j) \vee z_k \sim x_p \vee z_q$.*

Proof: Consider several cases. If $k < q$ then as $x_i \wedge y_j, z_k$ are maximally incomparable, then $x_i \wedge y_j \leq z_q$, so $(x_i \wedge y_j) \vee z_k \leq z_q \leq x_p \vee z_q$. If $q < k$ then as x_p, z_q are maximally incomparable, $x_p \leq z_k$, so $x_p \vee z_q \leq z_k \leq (x_i \wedge y_j) \vee z_k$. If $q = k$ and $i \leq p$ then $x_i \wedge y_j \leq x_p$, so $(x_i \wedge y_j) \vee z_k \leq x_p \vee z_q$. If $q = k$ and $p < i$ then as x_i, y_j are minimally incomparable, $x_p \leq y_j$, so $x_p \leq x_i \wedge y_j$ so $x_p \vee z_q \leq (x_i \wedge y_j) \vee z_k$. □

Lemma 24. *Assume (i) x_i, y_j are minimally incomparable in x, y, (ii) $x_i \wedge y_j$ is an element of com(x, y), (iii) $x_i \wedge y_j, z_k$ are maximally incomparable in com$(x, y), z$, and (iv) x_p, z_q are minimally incomparable in x, z. Then $(x_i \wedge y_j) \vee z_k \sim x_p \wedge z_q$.*

Proof: Consider several cases. If $p < i$ then as x_i, y_j are minimally incomparable, $x_p \leq y_j$, so $x_p \leq x_i \wedge y_j$, giving $x_p \wedge z_q \leq (x_i \wedge y_j) \vee z_k$. If $q \leq k$ then $z_q \leq z_k$, so $x_p \wedge z_q \leq (x_i \wedge y_j) \vee z_k$. Assume $i \leq p$ and $k < q$. As $i \leq p$ we have $x_i \leq x_p$, hence $x_i \wedge y_j \leq x_p$. Also, as $k < q$ and x_p, z_q are minimally incomparable, we have $z_k \leq x_p$. Combining these observations gives $(x_i \wedge y_j) \vee z_k \leq x_p$. As

$k < q$, $x_i \wedge y_j$ and z_k being maximally incomparable gives $x_i \wedge y_j \leq z_q$, hence $(x_i \wedge y_j) \vee z_k \leq z_q$. Therefore, $(x_i \wedge y_j) \vee z_k \leq x_p \wedge z_q$. □

Theorem 25. *$K(L)$ satisfies $com(com(com(x, y), z), com(x, z)) \approx 0$.*

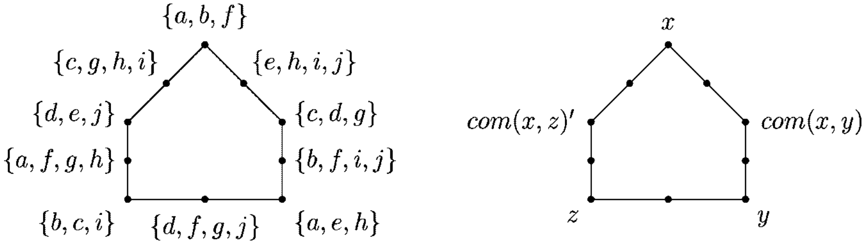
Proof: As the commutator of two elements equals 0 if, and only if, the elements commute, and two elements of $K(L)$ commute if, and only if, their elements form a chain, it is enough to show each element of the chain $com(com(x, y), z)$ is comparable to each element of the chain $com(x, z)$.

There are four possibilities for an element of $com(com(x, y), z)$, it must be of one of the forms (i) $(x_i \vee y_j) \vee z_k$, (ii) $(x_i \wedge y_j) \vee z_k$, (iii) $(x_i \vee y_j) \wedge z_k$ or (iv) $(x_i \wedge y_j) \wedge z_k$. Here there are further assumptions on the elements x_i, y_j being maximally incomparable in x, y if $x_i \vee y_j$ appears in the expression, and so forth. There are two possibilities for an element of $com(x, z)$, it must be of one of the forms (a) $x_p \vee z_q$ where x_p, z_q are maximally incomparable, or (b) $x_p \wedge z_q$ where x_p, z_q are minimally incomparable.

This gives a total of 8 possible combinations for an element of $com(com(x, y), z)$ and an element of $com(x, z)$. In the four lemmas earlier, we have considered case (i) and (a), case (i) and (b), case (ii) and (a), and case (ii) and (b). The other four cases follow by symmetry. □

Theorem 26. *The variety generated by the OMLs of the form $K(L)$ forms a proper subvariety of the variety of concrete OMLs.*

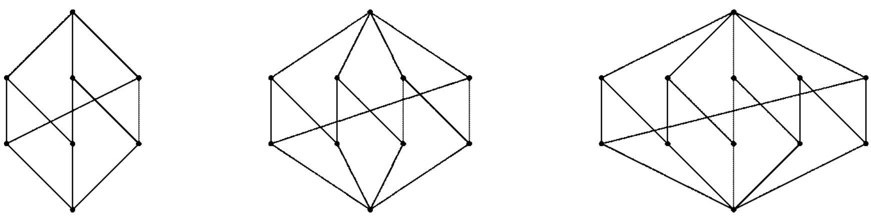
Proof: As every OML of the form $K(L)$ is concrete (Mayet and Navara, 1995), in view of Theorem 25 it is only necessary to produce a concrete OML that does not satisfy the identity $com(com(com(x, y), z), com(x, z)) \approx 0$. Consider the OML known as the 5-loop whose Greechie diagram (Kalmbach, 1983) is shown later (twice). For convenience we use L_5 to denote this OML.



The diagram at left indicates how L_5 can be realized as a collection of sets. Take the 10 subsets of $X = \{a, b, c, d, e, f, g, h, i, j\}$ in the diagram at left, their complements in X, as well as $\emptyset$ and X. The resulting collection of 20 sets is closed under set complementation and finite orthogonal joins, hence forms a class of sets. This class of sets is isomorphic to L_5, so L_5 is a concrete OML.

Consider the atoms x, y, z of L_5 shown in the diagram at right. The commutator $com(x, y)$ is the coatom that lies above both x, y and its orthocomplement is the atom shown in the diagram at right. Similar comments hold for $com(x, z)$. In any OML we have $com(p, q) = com(p'q)$. Therefore, $com(com(x, y), z)$ is the coatom lying above both $com(x, y)'$ and z, hence is equal to y'. This then shows that $com(com(com(x, y), z), com(x, z))$ is the coatom lying above both y and $com(x, z)'$, hence is equal to z'. In particular, L_5 does not satisfy the earlier identity. □

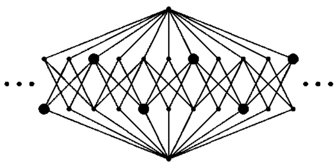
Remark For each $n \geq 3$ construct a bounded lattice C_{2n} by adding a top and bottom element to the poset known as an n-crown. The lattices C_6, C_8 and C_{10} are shown from left to right in the following diagram.



One can check that $K(C_6)$ is the OML known as the 6-loop L_6, that $K(C_8)$ is the 8-loop L_8, and so forth. In general, for any $n \geq 3$, the even-length loop L_{2n} is obtained by applying Kalmbach's construction to the bounded lattice C_{2n}. It is not difficult to convince oneself that an odd-length loop L_{2n+1} can not be obtained as $K(L)$ for any bounded lattice L. The results of this section have shown that L_5 not only is not of the form $K(L)$, but does not belong to the variety generated by the OMLs of the form $K(L)$.

Remark It seems plausible that the results of this section could be extended to show that for each $n \geq 2$, the odd-length loop L_{2n+1} does not belong to the variety V_K generated by OMLs of the form $K(L)$. This would have certain implications for the equational theory of this variety that we now describe.

Let μ be a non-principal ultrafilter over the natural numbers $\mathbb{N}$. Then the ultraproduct $\prod_\mu L_{2n+5}$ is the OML L formed by taking the horizontal sum of some large number (say κ) copies of the OML depicted in the following diagram, at left. The infinite cardinal κ depends on the ultrafilter μ. One can then see that L is obtained as $K(F)$ where F is the lattice obtained by taking κ disjoint copies of the poset shown at right (known as an infinite fence) and then adding a top and bottom to the result.



Then, if our results can be extended as supposed, we would have a family of concrete OMLs that do not belong to V_K, but whose ultraproduct does belong to V_K. It is well known (Chang and Keisler, 1990) that this implies the variety V_K cannot be defined by a finite set of identities, i.e. that V_K is not finitely based. Godowski (1981) showed that the variety of concrete OMLs is not finitely based. If our results can be extended as supposed, the fact that the OMLs L_{2n+5} are all concrete would further imply that V_K is not even finitely based with respect to the variety of concrete OMLs. This means that even given infinitely many identities required to define the variety of concrete OMLs, one requires infinitely many additional identities to define the variety V_K.

Remark We note that the variety V_K generated by the OMLs of the form $K(L)$ where L is a bounded lattice is, in fact, generated by the OMLs of the form $K(F)$ where F is a finite lattice. To see this, we must show that any identity $s \approx t$ that fails in some $K(L)$ with L a bounded lattice fails in some $K(F)$ with F a finite lattice.

Suppose that s, t are ortholattice terms, L is a bounded lattice, $x_1, \ldots, x_n$ are elements of $K(L)$, and $s^{K(L)}(x_1, \ldots, x_n) \neq t^{K(L)}(x_1, \ldots, x_n)$ where $s^{K(L)}$ and $t^{K(L)}$ are the interpretations of the terms s, t in the OML $K(L)$. Let S be the subset of L consisting of all elements of the chains $x_1, \ldots, x_n$ as well as all elements of L that occur at any stage in the evaluation of $s^{K(L)}(x_1, \ldots, x_n)$ and $t^{K(L)}(x_1, \ldots, x_n)$. Then S is a finite subset of L that we consider as a finite partial subalgebra of L. As the class of bounded lattices has the finite embedding property (Grätzer, 1979) there is a finite lattice F containing S as a partial subalgebra. From the description of joins, meets, and orthocomplementation given in (Harding, 1991), it follows that the evaluation of s and t at $x_1, \ldots, x_n$ in $K(L)$ agrees with the evaluation of these terms in $K(F)$, i.e. $s^{K(L)}(x_1, \ldots, x_n) = s^{K(F}(x_1, \ldots, x_n)$ and $t^{K(L)}(x_1, \ldots, x_n) = t^{K(F)}(x_1, \ldots, x_n)$. Therefore, the failure of $s \approx t$ in $K(L)$ produces a failure of this identity in $K(F)$.

This shows that the variety V_K is generated by its finite members. However, we do not know that V_K has a decidable equational theory (solvable free word problem) as we do not know that V_K can be defined by a recursively enumerable set of identities. It seems completely open whether the variety *Concrete* is generated by its finite members, or whether it has a decidable equational theory.

5. ATOMIC CONCRETE OMLS

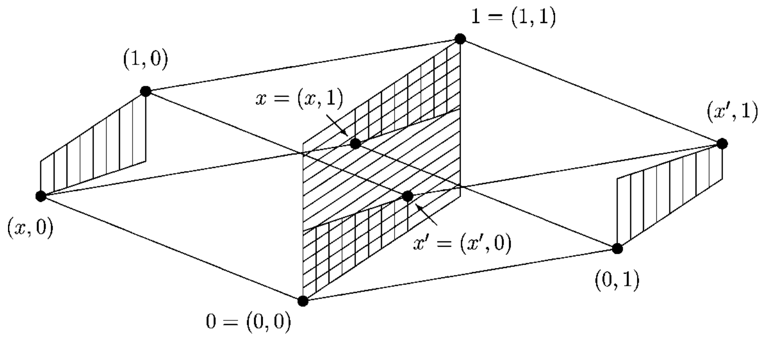
Using the coatom construction of Bruns and Kalmbach (1973), one can show that every OML can be embedded into an OML in which each element is a join of two or fewer atoms. The reader should consult Harding (2002) for a proof of this result, and for an account of its somewhat muddy history. In this section we show that this result, and its proof, remain valid in the setting of concrete OMLs. The key result is the following.

Lemma 27. *If L is a concrete OML and $x \in L$, then there is a concrete OML $L(x)$ such that (i) $L \leq L(x)$, (ii) each atom of L is an atom of $L(x)$, and (iii) x is a join of two or fewer atoms in $L(x)$.*

Proof: If x is either 0 or an atom of L set $L(x) = L$. Otherwise, let S be the section $[0, x'] \cup [x, 1]$ of L and use Greechie's "paste job" (Greechie, 1968) to paste L and $S \times 2$ along the isomorphic sections $[0, x'] \cup [x, 1]$ of L and $([0, x'] \times \{0\}) \cup ([x, 1] \times \{1\})$ of $S \times 2$. This produces an OML $L(x)$ that we next describe in somewhat informed terms (for a more precise treatment of the "paste job" see (Greechie, 1968)).

To form $L(x)$ take the union of L and $S \times 2$ and "identify" the intervals $[0, x']$ and $[x, 1]$ of L with the intervals $[0, x'] \times \{0\}$ and $[x, 1] \times \{1\}$ of $S \times 2$ respectively. The ordering of $L(x)$ is defined to be the union of the orderings on L and $S \times 2$, i.e. for $a, b \in L(x)$ we have $a \leq b$ if, and only if, either a, b both belong to L and a lies under b in L, or both a, b belong to $S \times 2$ and a lies under b in $S \times 2$. The orthocomplement of $L(x)$ is defined so that it extends the orthocomplementations on both L and $S \times 2$.

The situation is depicted in the following diagram, with the L shaded with diagonal lines and $S \times 2$ shaded with vertical lines. We note that (i) L is a subalgebra of $L(x)$, (ii) each atom of L is an atom of $L(x)$, and (iii) that $(0, 1)$ and $(x, 0)$ are atoms of $L(x)$ that join to the element $(x, 1)$ of $L(x)$ that is identified with the element x. Thus, it remains only to show that the OML $L(x)$ is concrete.



Recall that Godowski (1981) showed that an OML is concrete if, and only if, it has a full set of two-valued states. The crucial ingredient in showing that $L(x)$ is concrete is to show that two-valued states on L can be extended in certain ways to two-valued states on $L(x)$. In the following we assume $s : L \to \{0, 1\}$ is a two-valued state on L. We then define a map $s^1 : L(x) \to \{0, 1\}$ by setting

$$s^1(a) = \begin{cases} s(a) & \text{if} \quad a \in L \\ s(a_1) & \text{if} \quad a \in S \times 2 \text{ and } a = (a_1, a_2) \end{cases}$$

Further, if $s(x) = 1$ we define a map $s^2 : L(x) \to \{0, 1\}$ by setting

$$s^2(a) = \begin{cases} s(a) & \text{if} \quad a \in L \\ a_2 & \text{if} \quad a \in S \times 2 \text{ and } a = (a_1, a_2) \end{cases}$$

To see that s^1 and s^2 are well defined, suppose $a, b \in L$ with $a \leq x'$ and $x \leq b$, so that a is identified with $(a, 0)$ and b is identified with $(b, 1)$ in $L(x)$. The definition of s^1 provides directly that $s^1(a) = s^1((a, 0))$ and $s^1(b) = s^1((b, 1))$, thus s^1 is well defined. Also, if $s(x) = 1$, then $s(a) = 0$ and $s(b) = 1$, showing that $s^2(a) = s^2((a, 0))$ and $s^2(b) = s^2((b, 1))$, thus s^2 is well defined.

Clearly, s^1 and s^2 restrict to the state s on L, and one sees easily that s^1 and s^2 both restrict to states on $S \times 2$. But for any $p, q \in L(x)$ with $p \leq q'$ we have that either p, q both belong to L or they both belong to $S \times 2$. It then follows from the fact that s^1 and s^2 restrict to states on the subalgebras L and $S \times 2$ of $L(x)$ that s^1 and s^2 are states on $L(x)$.

As L is concrete, it has a full set of two-valued states. So for each $p, q \in L$ with $p \not\leq q$, there is a two-valued state $s_{p,q}$ on L with $s_{p,q}(p) = 1$ and $s_{p,q}(q) = 0$. To show $L(x)$ is concrete assume $a, b \in L(x)$ with $a \not\leq b$. By considering various cases we will show there is a two-valued state s on $L(x)$ with $s(a) = 1$ and $s(b) = 0$.

Case 1. $a, b \in L$.
Use $s^1_{a,b}$ *for* s.

Case 2. $a, b \in S \times 2$.

If $a_1 \not\leq b_1$ use $s^1_{a_1,b_1}$ for s. Otherwise, as $a_1 \leq b_1$ and $a \not\leq b$ we have $a_2 \not\leq b_2$, so $a_2 = 1$ and $b_2 = 0$. Choose a two-valued state on L taking value 1 at x, say $s_{x,x'}$, so we may form $s^2_{x,x'}$. Then use $s^2_{x,x'}$ for s.

Case 3. $a \in L - (S \times 2)$ *and* $b \in (S \times 2) - L$ *with* $b = (b_1, b_2)$.

Suppose $b_2 = 0$. As $a \notin S \times 2$ we have $a \not\leq x'$, so we may form $s_{a,x'}$. As $s_{a,x'}(x') = 0$ we have $s_{a,x'}(x) = 1$, and therefore we may form $s^2_{a,x'}$, and this serves as s. If $b_2 = 1$, then as $b \notin L$ we have $b_1 \not\geq x$, hence $b_1 \leq x'$. As shown earlier, $a \not\leq x'$, so $a \not\leq b_1$. We then use s^1_{a,b_1} for s.

Case 4. *$a \in (S \times 2) - L$ with $a = (a_1, a_2)$ and $b \in L - (S \times 2)$.*

If $a_2 = 0$ then as $a \notin L$ we have $a_1 \not\leq x'$, hence $x \leq a_1$. As $b \notin S \times 2$ we have $x \not\leq b$. Then use $s^1_{x,b}$ for s. If $a_2 = 1$, then as $a \notin L$ we have $x \not\leq a_1$, hence $a_1 \leq x'$. As before, $b \notin S \times 2$ gives $x \not\leq b$. Therefore, we may form $s_{x,b}$ and as $s_{x,b}(x) = 1$, we may form $s^2_{x,b}$ and use this for s.

We have shown that $L(x)$ has a full set of two-valued states, and therefore is a concrete OML. This completes the proof of our lemma. □

Theorem 28. *If L is a concrete OML, then there is a concrete OML $\hat{L}$ such that (i) $L \leq \hat{L}$, (ii) each atom of L is an atom of $\hat{L}$, and (iii) each element of $\hat{L}$ is a join of two or fewer atoms of $\hat{L}$.*

Proof: The proof follows that given by Harding (2002). One first shows that for any concrete OML L there is a concrete OML L^* such that (i) $L \leq L^*$, (ii) each atom of L is an atom of L^*, and (iii) each element of L is a join of two or fewer atoms of L^*. To accomplish this let $(x_\alpha)_\kappa$ be an indexing over a cardinal κ of the elements of L. Define recursively $L_0 = L$, $L_{\alpha+1} = L_\alpha(x_\alpha)$, and $L_\alpha = (\cup_{\beta<\alpha} L_\alpha)(x_\alpha)$ for α a limit ordinal. Then set $L^* = L_\kappa$. One then recursively defines a countable sequence of OMLs by setting $L^0 = L$ and $L^{n+1} = (L^n)^*$. Finally, define $\hat{L} = \cup_n L^n$. Then, as by Harding (2002), $\hat{L}$ is an OML with properties (i), (ii), and (iii). It remains only to show that $\hat{L}$ is concrete. But this follows as the union of a chain of concrete OMLs must be concrete, since the class of concrete OMLs form a variety (therefore, the failure of an identity in the union of a chain must occur already in some member of the chain). □

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